Roll No. ....

DD–2811

M. A./M. Sc. (Final) EXAMINATION, 2020

MATHEMATICS

(Optional)

Paper Fourth (*ii*)

(Wavelets)

Time : Three Hours

Maximum Marks : 100

Note : Attempt any *two* parts from each question. All questions carry equal marks.

Unit—I

1. (a) If the operator $P = P_{0,\in}$ defined by :

$$(P f)(x) \equiv \overline{S(x)}[S(x) f(x) \pm S(-x) f(-x)],$$

then show that P is idempotent, self-adjoint and an orthogonal projection.

(b) Define Multiresolution Analysis. Show that if $g \in L^2(\mathbf{R})$, then $\{g(.-k) : k \in Z\}$ is an orthonormal system if and only if :

$$\sum_{k \in Z} |\hat{g}(\xi + 2k\pi)|^2 = 1 \text{ for a.e. } \xi \in \mathbf{R}.$$

(c) Let r be a non-negative integer. Let ψ be a function in $C^r(\mathbf{R})$ such that :

$$|\psi(x)| \leq \frac{c}{(1+|x|)^{r+1+\epsilon}}$$

for some $\epsilon > 0$, and let $\psi^{(m)} \in L^{\infty}(\mathbf{R})$ for $m = 1, 2, \ldots, r$. If $\{\psi_{j,k} : j, k \in \mathbf{Z}\}$ is an orthonormal system in $L^2(\mathbf{R})$, then show that all moments of ψ upto order r zero; that is :

$$\int_{\mathbf{R}} x^m \psi(x)\, dx = 0$$

for all $m = 0, 1, 2, \ldots, r$.

Unit—II

2. (a) Suppose that $f \in L^2(\mathbf{R})$ and $\hat{f}$ has a support contained in $I = (a, b)$, where $b - a \leq 2^{-J}\pi$ and $I \cap [-\pi, \pi] = \phi$; then show that for all $j \in Z$:

$$(Q_j f)^{\wedge}(\xi) = \hat{f}(\xi)\left|\hat{\psi}(2^{-j}\xi)\right|^2$$

a.e. on I.

(b) If ψ is band-limited orthonormal wavelet such that $|\hat{\psi}|$ is continuous at 0, then show that $\hat{\psi} = 0$ a.e. in an open neighbourhood of the origin.

(c) If $f \in L^2(T)$, then show that :

$$\{f, Uf, \ldots, U^N f\},$$

where $U \equiv U_j$ is an orthonormal system if and only if :

$$\sum_{l \in Z} \left|F[f](n + 2^j l)\right|^2 = 2^{-j}$$

for $n = 0, 1, 2, \ldots, N = 2^j - 1$.

Unit—III

3. (a) If ψ is an orthonormal wavelet, then show that :

$$\psi(2^n\xi) = \sum_{j=1}^{\infty}\sum_{k \in Z} \hat{\psi}(2^n(\xi + 2k\pi))\, \overline{\hat{\psi}(2^j(\xi + 2k\pi))}$$

$$\hat{\psi}(2^j\xi)$$

a.e. for all $n \geq 1$.

(b) Let $\{v_j : j \geq 1\}$ be a family of vectors in a Hilbert space H such that :

(i) $\sum_{n=1}^{\infty} \|v_n\|^2 = c < \infty$

(ii) $v_n = \sum_{m=1}^{\infty} < v_n, v_m > v_m$ for all $n \geq 1$

Let $F = \overline{\text{span}\{v_j : j \geq 1\}}$, then show that :

$$\dim F = \sum_{j=1}^{\infty} \|v_j\|^2 = c.$$

(c) Define low-pass filter. Let $\mu_1, \mu_2, \ldots, \mu_n$ be 2π-periodic functions and set :

$$M_j = \sup_{\xi \in T}\left(|\mu_j(\xi)|^2 + |\mu_j(\xi + \pi|^2 \right),$$

then show that :

$$\int_{-2^n\pi}^{2^n\pi} \prod_{j=1}^{n} |\mu_j(2^{-j}\xi)|^2 \, d\xi < 2\pi M_1 \ldots M_n .$$

Unit—IV

4. (a) Define frame operator. For any $h \in L^2(\mathbf{R})$, if $Qh \in L^2(\mathrm{R})$ and $ph \in L^2(\mathbf{R})$, then show that :

(i) $R(Qh)(S, t) = S(Rh)(S, t) + \dfrac{1}{2\pi i}\dfrac{\partial}{\partial t}(Rh)(S, t)$

(ii) $R(Ph)(S, t) = -i\dfrac{\partial}{\partial S}(Rh)(S, t)$

(b) Suppose that :

$$g \in L^2(\mathbf{R})$$

and $\quad g_{m,n}(x) = e^{2\pi imx} g(x-n) \quad m, n \in Z$

If $\{g_{m,n} : m, n \in Z\}$ is a frame for $L^2(\mathbf{R})$, then show that either :

$$\int_R x^2 |g(x)|^2 \, dx = \infty$$

or

$$\int_R \xi^2 |\hat{g}(\xi)|^2 \, d\xi = \infty$$

(c) Show that when $\{Q_j : j \in J\}$ is a frame, f can be reconstructed from the coefficients $< f, Q_j >$ using the dual frame $\{\tilde{Q}_j : j \in J\}$ and that f is also superposition of Q'_j S with coefficients $< f, \tilde{Q}_j >$.

Unit—V

5. (a) If $N = 2^q$, $C_N = E_1E_2 \ldots E_q$, where each E_j is an $N \times N$ matrix such that each row has precisely two non-zero entries.

(b) Show that $\tilde{y}_k$ equals $\frac{1}{2}e^{\frac{\pi ik}{2N}}$ times the DCT coefficients $\alpha_k^{(N)}$ for the function f.

(c) Show that the sequence :

$$\{u_{j,k} : 1 \le k \le l_j - 1\}$$

is an orthonormal basis for E_j.